Math 155, ***Lecture Notes- Bonds*** **Name**____________

Section 9.7 *Taylor Polynomials and Approximations*

In this section, we will see how we can create special polynomial functions that can be used as approximations for other elementary functions. We will be using derivatives of an elementary function, we'll call this function $y=f(x)$, and we'll be expanding a polynomial function, we'll call this polynomial $y=P_n(x)$, about a fixed **center** point that we will call $(c,f(c))$. The original elementary function and this special polynomial will have behaviors that are extremely similar near the center point, $(c,f(c))$, and the more terms we use to expand our polynomial, the greater the accuracy we have for approximating $y=f(x)$, using $y=P_n(x)$.

Definitions of *n*th Taylor Polynomial and *n*th Maclaurin Polynomial

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the ***n*th Taylor polynomial for *f* at *c*.** If $c=0$, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

is also called the ***n*th Maclaurin polynomial for *f*.**

Ex. 1: Find the 6th degree Taylor Polynomial for $f(x)=\ln(1+x)$, centered at $c=0$.

$f'(x) = \frac{1}{1+x} = (x+1)^{-1}$

$f''(x) = -1\cdot(x+1)^{-2}\cdot(1) = -(x+1)^{-2}$

$f'''(x) = -2[-(x+1)^{-3}]\cdot 1 = 2(x+1)^{-3}$

$f^{IV}(x) = -3\cdot 2(x+1)^{-4}\cdot 1 = -6(x+1)^{-4}$

$f^{V}(x) = -4\cdot[-6(x+1)^{-5}]\cdot 1 = 24(x+1)^{-5}$

$f^{VI}(x) = -5\cdot 24(x+1)^{-6}\cdot 1 = -120(x+1)^{-6}$

$f(c) = f(0) = \ln[1+(0)] = \ln(1) = 0$

$f'(0) = (0+1)^{-1} = 1$

$f''(0) = -(0+1)^{-2} = -1$

$f'''(0) = 2(0+1)^{-3} = 2$

$f^{IV}(0) = -6(0+1)^{-4} = -6$

$f^{V}(0) = 24(0+1)^{-5} = 24$

$f^{VI}(0) = -120(0+1)^{-6} = -120$

$$P_6(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{IV}(0)}{4!}(x-0)^4 + \frac{f^{V}(0)}{5!}(x-0)^5 + \frac{f^{VI}(0)}{6!}(x-0)^6$$

Do not need to write these.

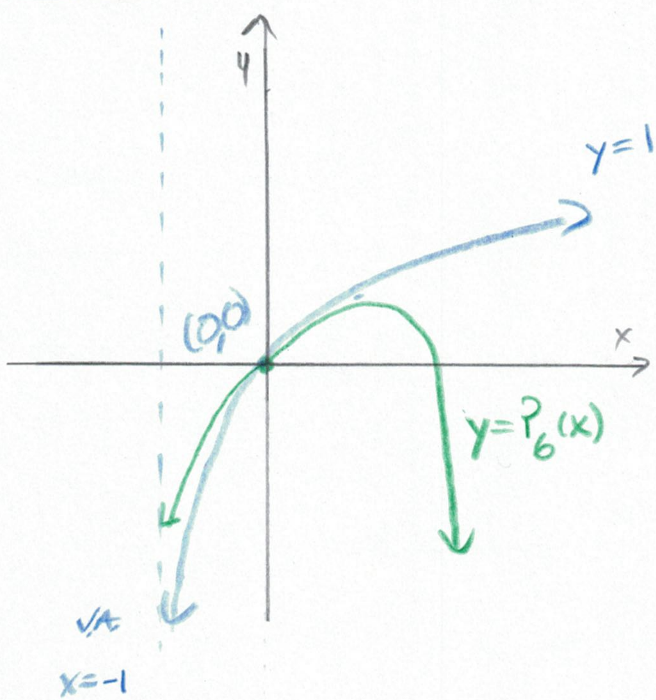
More Ex. 1:

$$P_6(x) = 0 + \frac{(1)}{1!}x^1 + \frac{(-1)}{2!}x^2 + \frac{(2)}{3!}x^3 + \frac{(-6)}{4!}x^4 + \frac{(24)}{5!}x^5 + \frac{(-120)}{6!}x^6$$

$$P_6(x) = x - \frac{1}{2}x^2 + \frac{2}{1\cdot2\cdot3}x^3 - \frac{6}{1\cdot2\cdot3\cdot4}x^4 + \frac{24}{1\cdot2\cdot3\cdot4\cdot5}x^5 - \frac{120}{1\cdot2\cdot3\cdot4\cdot5\cdot6}x^6$$

$$P_6(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6$$

← center is zero. Also seen as a Maclaurin Polynomial.



$P_6(x)$ approximates the behavior of $y = \ln(x+1)$ around $(0,0)$.

Ex. 2: Find the 6th degree Taylor Polynomial for $f(x) = \ln(1+x)$, centered at $c = 1$.

$f(x) = \ln(x+1)$

$f'(x) = (x+1)^{-1}$

$f''(x) = -(x+1)^{-2}$

$f'''(x) = 2(x+1)^{-3}$

$f^{IV}(x) = -6(x+1)^{-4}$

$f^{V}(x) = 24(x+1)^{-5}$

$f^{VI}(x) = -120(x+1)^{-6}$

$f(1) = \ln(1+1) = \ln(2)$

$f'(1) = (1+1)^{-1} = 2^{-1} = \frac{1}{2}$

$f''(1) = -(1+1)^{-2} = -2^{-2} = -\frac{1}{4}$

$f'''(1) = 2(1+1)^{-3} = 2 \cdot 2^{-3} = \frac{1}{4}$

$f^{IV}(1) = -6(1+1)^{-4} = -6 \cdot 2^{-4} = \frac{-6}{16} = \frac{-3}{8}$

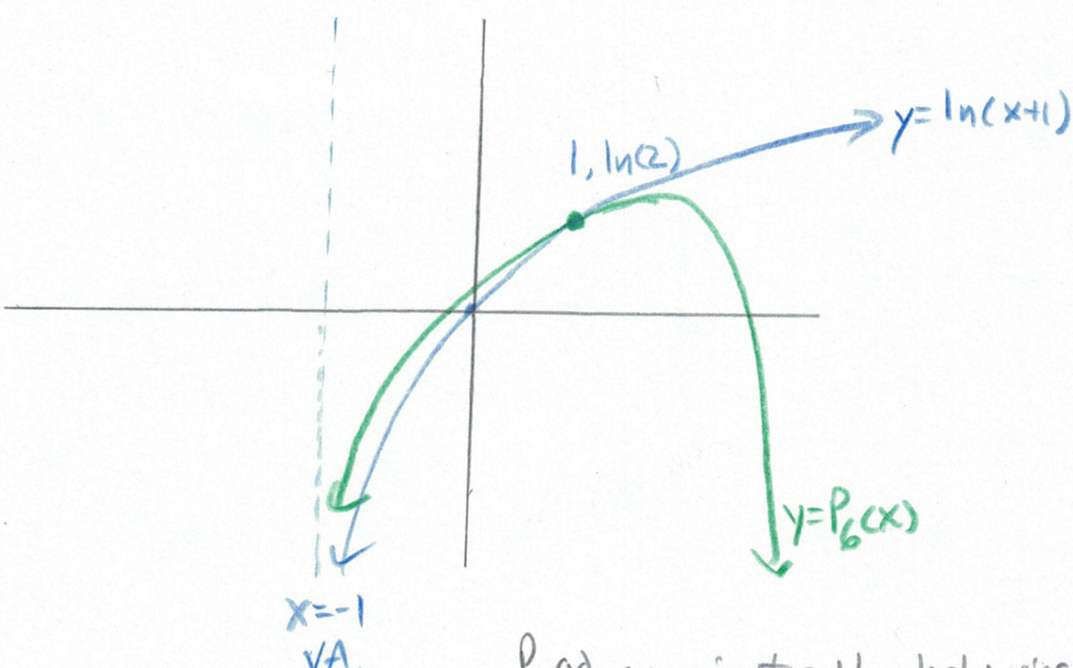
$f^{V}(1) = 24(1+1)^{-5} = 24 \cdot 2^{-5} = \frac{24}{32} = \frac{3}{4}$

$f^{VI}(1) = -120(1+1)^{-6} = -120 \cdot 2^{-6} = -\frac{120}{64} = \frac{-15}{8}$

$$P_6(x) = f(1) + \frac{f'(1)}{1!}(x-1)^1 + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{IV}(1)}{4!}(x-1)^4 + \frac{f^{V}(1)}{5!}(x-1)^5 + \frac{f^{VI}(1)}{6!}(x-1)^6$$

$$P_6(x) = \ln(2) + \frac{\left(\frac{1}{2}\right)}{1!}(x-1) + \frac{\left(\frac{-1}{4}\right)}{2!}(x-1)^2 + \frac{\left(\frac{1}{4}\right)}{3!}(x-1)^3 + \frac{\left(\frac{-3}{8}\right)}{4!}(x-1)^4 + \frac{\left(\frac{3}{4}\right)}{5!}(x-1)^5 + \frac{\left(\frac{-15}{8}\right)}{6!}(x-1)^6$$

$$P_6(x) = \ln(2) + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{24}(x-1)^3 - \frac{1}{64}(x-1)^4 + \frac{1}{160}(x-1)^5 - \frac{1}{384}(x-1)^6$$



F.Y.I.

$$\frac{\frac{-3}{8}}{4!} = \frac{-\cancel{3}}{8} \cdot \frac{1}{1 \cdot 2 \cdot \cancel{3} \cdot 4} = \frac{-1}{64}$$

$$\frac{\frac{3}{4}}{5!} = \frac{\cancel{3}}{4} \cdot \frac{1}{1 \cdot 2 \cdot \cancel{3} \cdot 4 \cdot 5} = \frac{1}{160}$$

$$\frac{\frac{-15}{8}}{6!} = \frac{-\cancel{15}}{8} \cdot \frac{1}{1 \cdot 2 \cdot \cancel{3} \cdot 4 \cdot \cancel{5} \cdot 6} = \frac{-1}{384}$$

$P_6(x)$ approximates the behavior of $y = \ln(x+1)$ around $(1, \ln(2))$.

Ex. 3: Find the 4^{th} degree Maclaurin Polynomial for $f(x)=e^{3x}$.

$c=0$

$$f(x)=e^{3x}$$
$$f'(x)=e^{3x}\cdot 3=3e^{3x}$$
$$f''(x)=3e^{3x}\cdot 3=9e^{3x}$$
$$f'''(x)=9e^{3x}\cdot 3=27e^{3x}$$
$$f^{IV}(x)=27e^{3x}\cdot 3=81e^{3x}$$

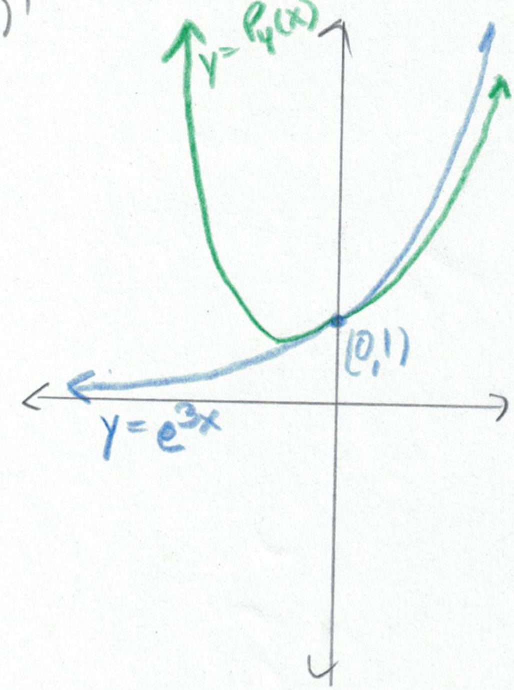
$$f(0)=e^{3(0)}=e^{0}=1$$
$$f'(0)=3e^{3(0)}=3\cdot e^{0}=3$$
$$f''(0)=9e^{3(0)}=9\cdot e^{0}=9$$
$$f'''(0)=27e^{3(0)}=27\cdot e^{0}=27$$
$$f^{IV}(0)=81e^{3(0)}=81\cdot e^{0}=81$$

$$P_4(x)=f(0)+\frac{f'(0)}{1!}(x-0)^1+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3+\frac{f^{IV}(0)}{4!}(x-0)^4$$

$$P_4(x)=1+\frac{3}{1!}(x)+\frac{9}{2!}(x)^2+\frac{27}{3!}(x)^3+\frac{81}{4!}(x)^4$$

$$P_4(x)=1+3x+\frac{9}{2}x^2+\frac{9}{2}x^3+\frac{27}{8}x^4$$

$P_4(x)$ approximates the behavior of $y=e^{3x}$ around $(0,1)$.



Ex. 4: Use a 4$^{\text{th}}$ degree Maclaurin Polynomial to approximate $\ln(1.05)$.

If we let $f(x) = \ln(x+1)$, then $f(0.05) = \ln(0.05+1) = \ln(1.05)$.

Also, we will be able to use $P_4(x) = f(0) + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{IV}(0)}{4!}(x-0)^4$ to approximate the behavior of $f(x) = \ln(x+1)$ near $c=0$. Hopefully, we can agree that 0.05 is near 0.

From the first example, we found the following:

$f(x) = \ln(x+1)$	$f(0) = \ln(0+1) = \ln(1) = 0$
$f'(x) = \frac{1}{x+1} = (x+1)^{-1}$	$f'(0) = (0+1)^{-1} = 1$
$f''(x) = -(x+1)^{-2}$	$f''(0) = -1(0+1)^{-2} = -1$
$f'''(x) = 2(x+1)^{-3}$	$f'''(0) = 2(0+1)^{-3} = 2$
$f^{IV}(x) = -6(x+1)^{-4}$	$f^{IV}(0) = -6(0+1)^{-4} = -6$

$$P_4(x) = 0 + \frac{(1)}{1!}(x) + \frac{(-1)}{2!}(x)^2 + \frac{(2)}{3!}(x)^3 + \frac{(-6)}{4!}(x)^4$$

$$P_4(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$$

Since $P_4(x) \approx \ln(x+1)$ centered at $c=0$,

we can see $P_4(0.05) \approx \ln(0.05+1)$

or $P_4(0.05) \approx \ln(1.05)$.

We have $P_4(0.05) = (0.05) - \frac{1}{2}(0.05)^2 + \frac{1}{3}(0.05)^3 - \frac{1}{4}(0.05)^4$

$P_4(0.05) \approx 0.048790104167$

From a calculator, $\ln(1.05) \approx 0.048790164169$.

So, $P_4(0.05)$ is a very close approximation of $\ln(1.05)$.

Is there a theorem about this?

THEOREM 9.19 Taylor's Theorem

If a function f is differentiable through order $n + 1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - c)^{n+1}.$$

NOTE: $\left|R_n(x)\right| \le \frac{|x-c|^{n+1}}{(n+1)!} \bullet \max\left|f^{n+1}(z)\right|$

where $\max\left|f^{n+1}(z)\right|$ is the maximum value of $f^{n+1}(z)$ between x and c.

"When applying Taylor's Theorem, you should not expect to be able to find the exact value of z. (If you could do this, approximation would not be necessary.) Rather, you try to find bounds for $f^{n+1}(z)$ from which you are able to tell how large the remainder $R_n(x)$ is."

Exact Value

Remainder

Vocabulary: $f(x) = P_n(x) + R_n(x)$

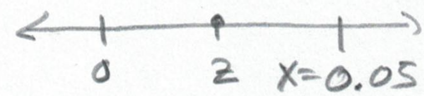
Taylor Approximation Value

$$Error = \left|R_n(x)\right| = \left|f(x) - P_n(x)\right|$$

Error Associated with Approximation

Ex. 5: How good was our $P_4(x)$ Maclaurin approximation of $\ln(1.05)$?

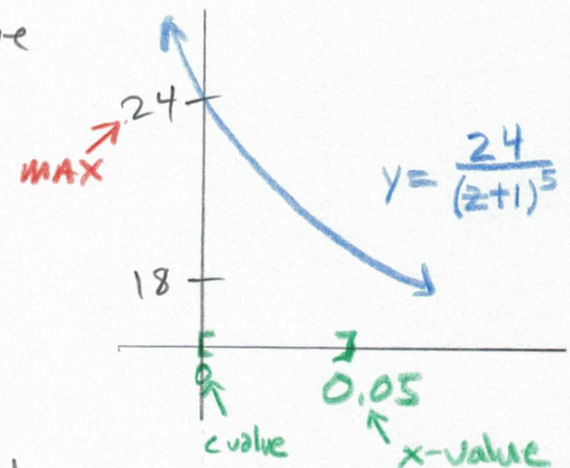
For $f(x) = \ln(x+1)$, we had $P_4(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, and we know $f(x) = P_4(x) + R_4(x)$. According to Taylor's Theorem, we have $R_4(x) = \frac{f^5(z)}{5!}(x-0)^5$, with



Since $f^{IV}(x) = -6(x+1)^{-4}$, we can see $f^{(5)}(x) = 24(x+1)^{-5}$, and $f^5(z) = 24(z+1)^{-5}$.

On the interval $[0, 0.05]$, we have

$$\max|f^{(n+1)}(z)| = \max|f^{(5)}(z)|$$
$$= \max\left|\frac{24}{(z+1)^5}\right|$$
$$= \left|\frac{24}{(0+1)^5}\right|$$
$$= 24$$



So, we have $|R_4(x)| \le \frac{|x-0|^5}{5!} \cdot \max|f^5(z)|$

$$|R_4(0.05)| \le \frac{|0.05-0|^5}{5!} \cdot 24$$
$$|R_4(0.05)| \le \frac{(0.05)^5 \cdot \cancel{24}}{\cancel{1 \cdot 2 \cdot 3 \cdot 4} \cdot 5}$$
$$|R_4(0.05)| \le \frac{(0.05)^5}{5}$$
$$|R_4(0.05)| \le 6.25 \times 10^{-8}$$

This means that our approximation of $\ln(1.05)$ using $P_4(0.05)$ was accurate to 8 decimal places.

Ex. 6: Estimate $e^{0.3}$ with an error less than 0.001. Determine the degree if this Maclaurin Polynomial.

$c=0$ ← centered at zero.

Let $f(x)=e^x$, and

$P_n(x) = f(0) + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \dots + \frac{f^n(0)}{n!}(x-0)^n$ is our MacLaurin Polynomial of degree n.

$f(x)=e^x$	$f(0)=e^0=1$
$f'(x)=e^x$	$f'(0)=e^0=1$
$f''(x)=e^x$	$f''(0)=e^0=1$
$f'''(x)=e^x$	$\vdots$
$\vdots$	
$f^n(x)=e^x$	$f^n(0)=e^0=1$

This gives us $P_n(x) = 1 + \frac{1}{1!}(x)^1 + \frac{1}{2!}(x)^2 + \frac{1}{3!}(x)^3 + \dots + \frac{1}{n!}(x)^n$.

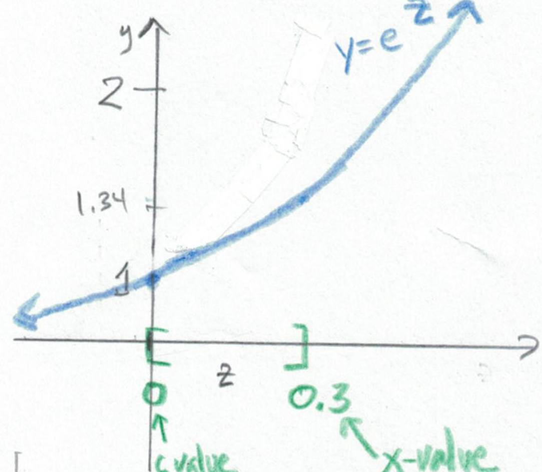
We know $f(x) = P_n(x) + R_n(x)$, with $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-0)^{n+1}$, according to Taylor's Theorem.

We have $|R_n(x)| = \frac{|x-0|^{n+1}}{(n+1)!} \cdot \max|f^{n+1}(z)|$

$|R_n(0.3)| \le \frac{(0.3)^{n+1}}{(n+1)!} \cdot \max|e^z|$

"Error" → $|R_n(0.3)| \le \frac{(0.3)^{n+1}}{(n+1)!} \cdot 2$

We want our "Error" to be less than 0.001, or $|R_n(0.3)| < 0.001$.



Since $y=e^z$ increases, we can estimate $e^{0.3} \le 2$ on $[0, 0.3]$.

So, $\max|f^{(n+1)}(z)| = \max|e^z| \le 2$.

More Ex. 6:

We can solve for n:

$$\frac{2(0.3)^{n+1}}{(n+1)!} < 0.001$$

$$\frac{2\left(\frac{3}{10}\right)^{n+1}}{(n+1)!} < \frac{1}{1000}$$

$$2 \cdot 1000 < [(n+1)!]\left(\frac{10}{3}\right)^{n+1}$$

$$2{,}000 < [(n+1)!] \cdot \left(\frac{10}{3}\right)^{n+1}$$

If $n=3$,

$$2{,}000 < [4!] \cdot \left(\frac{10}{3}\right)^{4}$$

$$2{,}000 < 24 \cdot \left(\frac{10{,}000}{81}\right)$$

$$2{,}000 < 2{,}962.96 \quad \text{, TRUE!}$$

This means that a Maclaurin Polynomial of degree 3 will approximate $e^{0.3}$ within 0.001.

So, we have $P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3$

$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3.$$

$$P_3(0.3) = 1 + (0.3) + \frac{1}{2}(0.3)^2 + \frac{1}{6}(0.3)^3$$

$$P_3(0.3) = 1.3495$$

$$P_3(0.3) \approx e^{0.3}$$

$$1.3495 \approx e^{0.3}$$

From a calculator, $e^{0.3} \approx 1.34985880758$.

We can see that $P_3(0.3)$ is within 0.001 of $e^{0.3}$.

Ex. 7: Determine the degree of the Maclaurin Polynomial required for the error in the approximation of $\cos(0.1)$ to be less than 0.001. $(c=0) \leftarrow$ centered at zero

Let $f(x) = \cos(x)$, and

$$P_n(x) = f(0) + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \dots + \frac{f^n(0)}{n!}(x-0)^n$$ is our

MacLaurin Polynomial of degree n.

$f(x) = \cos(x)$	$f(0) = \cos(0) = 1$
$f'(x) = -\sin(x)$	$f'(0) = -\sin(0) = 0$
$f''(x) = -\cos(x)$	$f''(0) = -\cos(0) = -1$
$f'''(x) = \sin(x)$	$f'''(0) = \sin(0) = 0$
$f^{iv}(x) = \cos(x)$	$f^{iv}(0) = \cos(0) = 1$
$f^{v}(x) = -\sin(x)$	$f^{v}(0) = -\sin(0) = 0$
$f^{vi}(x) = -\cos(x)$	$f^{vi}(0) = -\cos(0) = -1$
$\downarrow$ etc...	$\downarrow$ etc...

This gives us $P_n(x) = 1 + \frac{(0)}{1!}(x) + \frac{(-1)}{2!}(x)^2 + \frac{(0)}{3!}(x)^3 + \frac{(1)}{4!}(x)^4 + \dots$

$$P_n(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \dots$$

$$P_n(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \dots + \frac{(-1)^n x^{2n}}{(2n)!}.$$

We know $f(x) = P_n(x) + R_n(x)$, with $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-0)^{n+1}$.

That is, $R_n(x) = \frac{f^{(n+1)}(z)\, x^{n+1}}{(n+1)!}$. $f^{(n+1)}(z)$ is either $\pm\sin(z)$ or $\pm\cos(z)$, depending on n.

When considering $\max|f^{(n+1)}(z)|$, we have $\max|\pm\sin(z)| = 1$, or we have $\max|\pm\cos(z)| = 1$. This tells us that $\max|f^{(n+1)}(z)| = 1$.

More Ex. 7:

We have $\left|R_n(x)\right| = \frac{|x-0|^{n+1}}{(n+1)!} \cdot \max\left|f^{(n+1)}(z)\right|$, and

$$\left|R_n(0.1)\right| \le \frac{(0.1)^{n+1}}{(n+1)!} \cdot 1$$

$$\left|R_n(0.1)\right| \le \frac{(0.1)^{n+1}}{(n+1)!}$$

We want our "Error" to be less than 0.001, or

$$\left|R_n(0.1)\right| < 0.001 .$$

We can solve for n:

$$\frac{(0.1)^{n+1}}{(n+1)!} < 0.001$$

$$\frac{\left(\frac{1}{10}\right)^{n+1}}{(n+1)!} < \frac{1}{1{,}000}$$

$$1{,}000 < [(n+1)!] \cdot \left(\frac{10}{1}\right)^{n+1}$$

$$1{,}000 < [(n+1)!] \cdot (10)^{n+1}$$

If $n=2$:

$$1{,}000 < [3!] \cdot 10^3$$

$$1{,}000 < 6 \cdot 1{,}000$$

$$1{,}000 < 6{,}000 \text{, TRUE!}$$

This means that a MacLaurin Polynomial of degree 2 will approximate $\cos(0.1)$ with an error of less than 0.001. So, we have $P_2(x) = 1 - \frac{1}{2}x^2$ that can approximate $\cos(0.1)$.

$$P_2(0.1) = 1 - \frac{1}{2}(0.1)^2$$

$$P_2(0.1) = 0.995$$

$$P_2(0.1) \approx \cos(0.1)$$

$$0.995 \approx \cos(0.1)$$

From a calculator, $\cos(0.1) \approx 0.995004165278$. We can see $P_2(0.1)$ is within 0.001 of $\cos(0.1)$.